Transmission Line Theory

Before we jump into transmission lines, we need to make an important connection. In Chapter 1, everything we talked about lived in either:

- Free Space
- Uniform Materials

Waves spread out, fill space, and follow Maxwell's equations in their full vector form. That's great for understanding how electromagnetic waves behave in the wild, but that's not how real RF systems move energy from point A to point B. Real RF hardware doesn't let waves float around in space. They are **guided**. To picture this guided wave for a transmission line (a structure that carries high-frequency electrical signals from one point to another without losing too much energy and/or distorting the signal), imagine yelling into a canyon versus yelling into a tube. In a canyon, your voice spreads out quickly and attenuates rapidly over distance. In a tube, your voice travels further and stays intact. The transmission line is like that pipe - it contains and guides the electromagnetic wave, preventing it from leaking or reflecting unpredictably

The moment you introduce two conductors, the electromagnetic fields are no longer free to do whatever they want. The metal surfaces impose boundary conditions that trap the fields in a very specific way. The result is a guided wave - an electromagnetic wave that is forced to follow a path defined entirely by the geometry of the conductors.

What we call "voltage" and "current" on a transmission line are just the electromagnetic fields expressed in circuit language.

The electric field between the conductors becomes voltage.

The magnetic field wrapping around the conductors becomes current.

The Poynting vector - the flow of electromagnetic energy - flows down the line.

So even though transmission lines look like a circuit-theory object, everything happening on them is still pure Maxwell under the hood. In fact, the Telegrapher's Equations we're about to derive come directly from Maxwell's equations applied to this guided structure. They are simply Maxwell's equations in a one-dimensional, engineering-friendly form.

Circuit Models for Transmission Lines

Let's go back to the topic of wavelength. As mentioned before, the wavelength for signals in analogue devices is massive and enables us to treat each element (resistor, capacitor, or inductor) as lumped elements. At radio frequencies, this electrical length is really small, so we have to treat the element as a distributed element.

Let's pause. What's a lumped element? What's a distributed element? Why do we have to treat high frequency signals as distributed elements instead of lumped elements? What's going on here?

In analogue electrical circuits where the boundary is set by the electrical wavelength relative to wavelength, lumped elements are idealized components like resistors, capacitors, and inductors that we assume are concentrated at a single point in space - they don't take up electrical space in the circuit **in terms of their electrical behavior** since the wavelength is so large. This doesn't mean that one of these components doesn't take up space on a circuit board. Of course it does! But as far as the signal itself is concerned, these elements don't take up any space at all. If the signal wavelength is much larger than the components size (say a capacitor 3mm long), the entire signal "sees" the component at one instant in time. For example, say you have a wavelength of 30 kilometers - a very big wave. For a 3mm capacitor, it appears extraordinarily small compared to the 30km wave - the physical size of the component is much smaller than the wavelength of the signal it's handling. Technically speaking, the capacitor takes up 0.00001% of the wavelength. The physical length of the 3mm capacitor is negligible compared to the 30km wave. At these lower frequencies, we can lump all of these components together to determine a certain electrical output since they appear so small compared to the signals' wavelength. This is why they are called *lumped elements.*

At high frequencies, the lumped element model starts to break down. Capacitors start to radiate or act like an antenna, a wire behaves like an inductor or transmission line, and parasitics (stray inductance and capacitance) become significant. This is when we need to use *distributed elements*. Let's take the same 3mm capacitor, but instead of a 30km wave, the wave is 30mm - much smaller. When comparing the 3mm capacitor to the 30mm wave, the capacitor is 10% of the wavelength. Now, the physical length of the 3mm capacitor must be taken into consideration. It's not nothing. The circuit is no longer a collection of "points" connected by "ideal wires." Instead, it's a structure - where geometry, spacing, and material all influence the behavior of the signal.

In short:

At low frequencies, components are points.

At high frequencies, components are paths

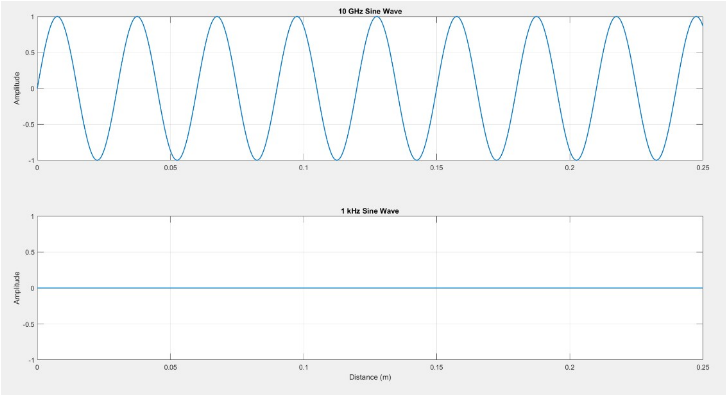
This shift is why RF and microwave engineers think of distributed structures - they draw filters, transformers, and matching networks with shapes and segments of line, not just RLC components.

The electrical length for microwave devices proves to be quite small and forces the designer to treat each element as a distributed element. In a microwave circuit, the elements are distributed in a certain way (over distance) to determine the voltage and current over various points in the circuit to determine the output. In terms of voltages and current for analogue signals, the amplitude and phase of the voltage and current do not vary too much over the length of the circuit, but in terms of voltages and current for microwave signals, the voltage and current amplitude and phase change over the distance of the circuit.

The Imaginary RF Probe

Let's compare a 1kHz analogue signal and 10GHz microwave signal - similar to what we did in *Chapter 1*, but this time, let's probe the circuit and its respective signal with an imaginary RF probe to determine the voltage at points *along* the circuit in a time snapshot. I say this is an imaginary RF probe, because this is not a proper or valid way to measure RF circuits. This is more of an exercise to illustrate why microwave signals are different than analogue/low frequency signals in terms of a circuit representation.

Let's look at the same 1kHz and 10GHz wave in *Chapter 1*:



10GHz vs. 1kHz Wave

The below signal represents the 1kHz analogue signal and the above signal represents the 10GHz microwave signal.

The y-axis is the real voltage amplitude and on the x-axis, the length of the circuit circuit. Let's say the length of the circuit is 0.25 meters long. If I were to take an imaginary radio frequency probe and probe any point on the circuit (.05m, 0.1cm, 0.2m, etc.), I would get a constant voltage readout of 0V at all points due to the amplitude and phase remaining constant across the distance of the circuit at one fixed moment in time. Similarly, if I were to take the same imaginary RF probe and probe the voltage of the microwave signal at various points on the circuit at one fixed moment in time, I would not get a constant voltage readout. If I probed the circuit at .05m, I would get a voltage readout of around -0.8V. If I probed the circuit at 0.25cm, the voltage readout would be around 0.9V.

This means the circuit parameters for the microwave signal (resistance, inductance, capacitance, and conductance) need to be represented in terms of distance. Resistance in analogue units is in terms of ohms, but for microwave units, it's in ohms/meter. Inductance in analogue units is in terms of henry's, but for microwave units, it's in henry's/meter. Conductance in analogue units is in terms of siemens, but for microwave units, it's in siemens/meter. Capacitance in analogue units is in terms of faraday's, but for microwave units, it's in faraday's/meter.

In a lumped microwave schematic, each element is represented in terms of a change in distance Δz. Similarly, the voltage and current vary with a change in distance Δz which is why Δz is added to the varying voltage and current definitions at the output of the circuit. Looking at the circuit elements a little further shows that RΔz is the resistance due to the conductivity between the two conductors, LΔz represents the inductance of the two conductors, shunt conductance GΔz represents the dielectric loss between the two conductors, and shunt capacitance CΔz is the capacitance between these two conductors.

2.1 The Lossy Transmission Line and the Telegrapher Equations

Wave propagation for voltages and currents are defined by performing Kirchhoff's Current Law (KCL) and Kirchhoff's Voltage Law (KVL on the components of the lumped model equivalent of the transmission line. These solutions will then pave the way to deriving the traveling wave solutions of lossless transmission lines. We won't perform KCL/KVL, but what is important is *why* we perform KCL/KVL on any circuit - particularly RF circuits. To start, it helps derive the *telegrapher equations*, which are fundamental to understanding wave behavior in transmission lines. The lumped model above divides a transmission line into small segments, each with series impedance and shunt admittance.

Series resistance (RΔz) and inductance (LΔz) model energy loss and magnetic field storage.

Shunt conductance (GΔz) and capacitance (CΔz) model leakage and electric field storage

After performing KCL and KVL the sinusoidal steady-state condition can be derived from the time domain solution output of KCL on the distributed circuit.

By simplifying the time domain solution output, the telegrapher equations are formally derived and defined in the frequency domain:

$$\frac{dV(z)}{dz} = -(R + j\omega L)\ I(z) \qquad (2.1.01)$$

$$\frac{dI(z)}{dz} = -(G + j\omega C)\ V(z) \qquad (2.1.02)$$

Let's break down each of these equations to understand what they're describing.

The first equation describes how voltage changes along the line based on the current and the line's resistance, R, and inductance, L.

The R term accounts for energy loss (like heat) as current flows.

The $j\omega L$ term accounts for the magnetic field buildup due to inductance

In plain speak, if current is flowing through a long, lossy wire, the voltage will drop as you move along the wire due to the resistive losses (energy loss) and inductive impedance (stored but reactive energy).

The second equation describes how current changes along the line based on the voltage and the lines conductance, G, and capacitance, C.

The term, G, models leakage through the dielectric (current leaking from wire to ground)

The $j\omega C$ models electric field storage (capacitive effect between conductors)

If voltage exists between two conductors, you get a charging effect - more voltage leads to more current flowing, not necessarily forward, but due to the displacement current (capacitive charging) and leakage.

Now, let's look at the big picture. Imagine a garden hose as your transmission line.

Voltage is "like pressure"

Current is like "water flow". Not in the sense where electrons move from point A to point B, but in the sense the flow is associated with a traveling pressure wave.

Inductance is like the hose resisting rapid changes in flow (inertia)
Capacitance is like the hose elastically storing pressure energy – similar to a balloon segment that compresses and releases.
Resistance and conductance are like leaks in the hose - they dissipate energy.

As with all analogies, this model is really intended to build *intuition* – something we will constantly try to build upon throughout the book. Imagining these waves alone without some sort of intuition and purely mathematical reasoning is extraordinarily difficult to do off the rip. Hence, this is where some analogies come into play. For this one in particular, it should not be interpreted as a literal description of electron motion.

Solving the Telegrapher equations simultaneously gives you the voltage and current wave equations. Now, instead of electric and magnetic field wave equations which were covered in Chapter 1, these equations describe how voltage and current propagate as waves along a transmission line.

Let's simultaneously solve these equations to achieve the voltage and current wave equation. First, begin with the voltage equation and partially differentiate with respect to z.

$$\frac{dV(z)}{dz} = -(R + j\omega L)\ I(z)$$

$$\frac{d^2V(z)}{dz^2} = -(R + j\omega L)\ \frac{dI(z)}{dz}$$

Going back to the current equation, we actually know what $\frac{dI(z)}{dz}$ is equal to. Let's plug the current equation into the now 2nd order differential equation for voltage. This is what it means to simultaneously solve these equations. You use one equation to solve for the other - very standard practice for deriving equations such as these.

$$\frac{dI(z)}{dz} = -(G + j\omega C)\ V(z)$$

$$\frac{d^2V(z)}{dz^2} = -(R + j\omega L)\ \frac{dI(z)}{dz}$$

$$\frac{d^2V(z)}{dz^2} = -(R + j\omega L)\ [(-(G + j\omega C)\ V(z))]$$

$$\frac{d^2V(z)}{dz^2} = (R + j\omega L)(G + j\omega C)\ V(z)$$

Similar to our electric field equation where permittivity and permeability define the propagation constant, so does a lossy transmission line with delta resistance, inductance, capacitance, and conductance.

$$\gamma^2 = (R + j\omega L)(G + j\omega C)$$
$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$

Very similar to when we solved for the propagation constant in Chapter 1 and plugged back into the equation.

$$\frac{d^2 V(z)}{dz^2} = (R + j\omega L)(G + j\omega C)\ V(z)$$
$$\frac{d^2 V(z)}{dz^2} = \gamma^2 V(z)$$
$$\frac{d^2 V(z)}{dz^2} - \ \gamma^2\ V(z) = 0$$

This should look VERY familiar if you have been following the derivations closely. This is another second-order differential equation! This equation specifically is the second order differential equation for a lossy transmission line.

Now, let's solve for the current wave equation using similar steps to solve the voltage wave equation. First, start with the current equation derived from KCL on the lumped transmission line circuit model and differentiate both sides with respect to z.

$$\frac{dI(z)}{dz} = -(G + j\omega C)\ V(z)$$
$$\frac{d^2 I(z)}{dz^2} = -(G + j\omega C)\frac{dV(z)}{dz}$$

Once again, we know what $\frac{dV(z)}{dz}$ is equal to, so let's plug that back into the above equation to simplify further.

$$\frac{d^2 I(z)}{dz^2} = -(G + j\omega C)\frac{dV(z)}{dz}$$

$$\frac{dV(z)}{dz} = -(R + j\omega L)\ I(z)$$

$$\frac{d^2 I(z)}{dz^2} = -(G + j\omega C)[(-(R + j\omega L)\ I(z))]$$

$$\frac{d^2 I(z)}{dz^2} = (R + j\omega L)(G + j\omega C)\ I(z)$$

Since we have already established what the propagation constant is, let's substitute the constant again to find the current wave equation for a transmission line.

$$\frac{d^2 I(z)}{dz^2} = (R + j\omega L)(G + j\omega C)\ I(z)$$

$$\frac{d^2 I(z)}{dz^2} = \gamma^2\ I(z)$$

$$\frac{d^2 I(z)}{dz^2} - \gamma^2\ I(z) = 0$$

After derivation, the voltage and current equations are given as:

$$\frac{d^2 V(z)}{dz^2} - \gamma^2\ V(z) = 0 \qquad (2.1.03)$$

$$\frac{d^2 I(z)}{dz^2} - \gamma^2\ I(z) = 0 \qquad (2.1.04)$$

If you've been paying attention so far, the general equations should seem obvious. Very similar to the general solution of the plane wave electric field wave equation, the general solution of the voltage and current wave equations follows suit in the following manner:

$$V(z) = V_o^+ e^{-\gamma z} + V_o^- e^{+\gamma z} \quad (2.1.05)$$

$$I(z) = I_o^+ e^{-\gamma z} + I_o^- e^{+\gamma z} \quad (2.1.06)$$

The first equation represents a voltage wave traveling in both directions (-z and +z). The first term, $V_o^+ e^{-\gamma z}$, represents the voltage wave traveling in the positive z-direction (a forward voltage traveling wave) and the second term, $V_o^- e^{+\gamma z}$, represents the voltage wave traveling in the negative z-direction (a reverse voltage traveling wave). Since this overall voltage wave is propagating on a lossy transmission line, the forward and reverse voltage waves will decay as they propagate due to the attenuation constant, αz. Remember, for lossy media including transmission lines, the propagation constant is not purely real.

Unless you're extraordinarily gifted at visualizing the obscure phenomena of wave propagation (in this case, voltage and current), it might be helpful describing these general solutions with intuitive examples as we've been doing so far. Hopefully, most people can relate to doing this when they were a kid - either with a rope, a chain, or a piece of string.

Imagine you're at a beach with a ten foot rope tied to a post and you give the rope a singular snap. This snap can be thought of as the source of the signal. As the rope "wave" travels to the post, the crest (amplitude) gets smaller. This forward traveling wave to the post represents $V_o^+ e^{-\gamma z}$ - a forward traveling voltage wave. Attenuation, which is part of the complex propagation constant, γ, causes the wave to become smaller as it approaches the post if the rope has friction. Now, as soon as that rope "wave" hits the post, it will reflect off the post, come back, and will also get smaller as it travels back towards your hand. This reflected traveling rope wave represents $V_o^- e^{+\gamma z}$ and similar to the forward traveling rope "wave", attenuation causes the amplitude/crest of the wave to decay.

Once again, make note that the propagation constant, γ, is utilized instead of β (or k for lossless transmission lines). This is because the complex propagation constant is defined as γ in lossy mediums (as mentioned before with plane waves and guided waves).

We have so far defined the traveling wave equations of Fig. b) by incorporating the resistance, inductance, conductance, and capacitance of the lumped model. The traveling wave solutions to these equations can be written as:

Notice the close similarity between these sets of general solutions for propagation:

Voltage and Current Traveling Wave:

$$V(z) = V_o^+ e^{-\gamma z} + V_o^- e^{+\gamma z}$$
$$I(z) = I_o^+ e^{-\gamma z} + I_o^- e^{+\gamma z}$$

Electric Field Traveling Plane Wave

$$E(z) = E_o^+ e^{-\gamma z} + E_o^- e^{+\gamma z}$$

Noticing the stark similarities between these two sets of equations is extremely important, because this is essentially the beginning of merging field theory and circuit theory for radio frequencies. For the voltage and current traveling waves, the propagation constant γ is

complex due to the lossy nature of the medium. Instead of an electric field plane wave propagating in the z-directions, we have voltage and current waves propagating in the z-directions. The voltage and current traveling waves include inductance, resistance, capacitance, and conductance
components which are defined by the complex propagation constant γ.

While Maxwell's equations give rise to electric and magnetic field plane waves propagating in the z-direction, an analogous concept in circuit theory emerges: voltage and current waves propagating along the same direction. As denoted before, we can deduce that the $e^{-\gamma z}$ term represents the traveling wave propagating in the forward +z direction and the $e^{+\gamma z}$ term represents the traveling wave propagating in the opposite -z direction for both the voltage and current waves.

The reader might be thinking, "Okay, we have voltage, we have current, what about impedance in terms of these waves?" This leads us to one of the most fundamental concepts in RF engineering and design: characteristic impedance. It is defined as the ratio of the amplitude of the voltage wave to the amplitude of the current wave on a transmission line. In practice, this is often treated as a real constant-typically 50Ω - because the losses are assumed to be minimal, making the imaginary components negligible.

To better describe characteristic impedance, let's go back to the intuitive example with water flowing in a pipe. Now, let's grab a piston and put it on one end of the pipe and give it a singular pump. As mentioned before:

Voltage is "like pressure"

Current is like "water flow". Not in the sense where electrons move from point A to point B, but in the sense the flow s associated with a traveling pressure wave.

Characteristic impedance = how much pressure (voltage) is needed to get a certain flow rate (current) for a traveling wave in a uniform pipe.

Suppose the pipe goes on forever and is completely uniform on the inside. When you push the piston, the pressure wave travels down the pipe and the water flows. The relationship between how hard you have to push (pressure = voltage) and the amplitude of the flow associated with the traveling pressure wave (flow = current) depends only on the pipe's properties - its diameter, stiffness, and the viscosity of the water.

To be more technical for my RF Engineering readers out there:

Voltage is the electric potential difference between conductors

Current is the *motion* of charge in the conductors that accompanies a propagating electromagnetic wave.

The characteristic impedance is the pressure-to-flow ratio of a traveling wave in a uniform line.

Consider this: imagine your goal is to get **all** the water to propagate smoothly to one end of a pipe. Sounds simple enough. If the pipe is uniform, the water flows without issue.

But now imagine that near the end, the pipe narrows or you partially close it off. What happens? Some of the water still flows through, but part of the wave reflects and comes back toward the source. This reflected wave disrupts the steady movement of water and causes interference or turbulence in the pipe.

This is exactly what happens in an electrical transmission line when the impedance changes at the load. A mismatch causes part of the voltage and current wave to reflect, which interferes with the forward wave - just like the backward flow of water in the pipe.